24-Feb L9

Monday, February 24, 2025 10:46 AM

Section 2.3 cont'd

recall: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, etc.

given: $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

$B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$

$$B(A\vec{e}_1) = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6(1)+7(3) \\ 8(1)+9(3) \end{bmatrix} = \begin{bmatrix} 27 \\ 35 \end{bmatrix}$$

use 1^{ST} col. of A

$$B(A\vec{e}_2) = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 6(2)+7(5) \\ 8(2)+9(5) \end{bmatrix} = \begin{bmatrix} 12+35 \\ 16+45 \end{bmatrix} = \begin{bmatrix} 47 \\ 61 \end{bmatrix}$$

then $T(\vec{x}) = B(A\vec{x})$ is $\begin{bmatrix} | & | \\ T(\vec{e}_1) & T(e_2) \\ | & | \end{bmatrix}$

$$= \begin{bmatrix} 27 & 47 \\ 35 & 61 \end{bmatrix}$$

expand to include larger matrices:

$$\begin{tikzcd} & \vec{z} \text{ in } \mathbb{R}^n \\ & & \vec{y} \text{ in } \mathbb{R}^p \arrow[ul, "\vec{z} = B\vec{y}"'] \\ & \vec{x} \text{ in } \mathbb{R}^m \arrow[uu] \arrow[ur, "\vec{y} = A\vec{x}"'] \end{tikzcd}$$

then $\vec{z} = B(A\vec{x})$ where BA is an $n \times m$ matrix

$= (BA)\vec{x}$ because $\mathbb{R}^m \to \mathbb{R}^p \to \mathbb{R}^n$

product of matrices

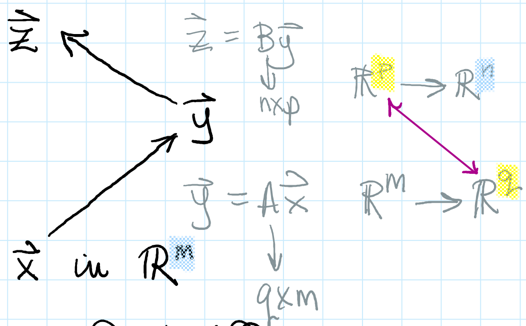
Doesn't Always Work:

suppose B is an $n \times p$ matrix where $p \neq q$

A is a $q \times m$ matrix

then $\vec{z} = B\vec{y}$ and $\vec{y} = A\vec{x}$ cannot be composed

$\vec{y} = A\vec{x}$



mismatch of dimensions

<u>rule of thumb:</u> BA is defined iff $p = q$
then BA is an $n \times m$ matrix

Matrix Multiplication

let $B = n \times p$ matrix
$A = p \times m$ matrix with colomns $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$

then $BA = \begin{bmatrix} | & | & & | \\ B\vec{v}_1 & B\vec{v}_2 & \cdots & B\vec{v}_m \\ | & | & & | \end{bmatrix}$ then combine resulting vectors

refer to prior example... $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$ $A = \begin{array}{c} \begin{matrix} \vec{v}_1 & \vec{v}_2 \end{matrix} \\ \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \end{array}$

then $\underset{2\times 2}{B}\underset{2\times 2}{A} = \begin{bmatrix} B\vec{v}_1 & B\vec{v}_2 \end{bmatrix}$

$$= \left[\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} \right]$$

refer to previous computations...

$$= \begin{bmatrix} 27 & 47 \\ 35 & 61 \end{bmatrix}$$

is matrix multiplication commutative? answer: not necessarily...

$AB = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$ ← Do: compute resulting vectors

$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 6+16 \\ 18+40 \end{bmatrix} = \begin{bmatrix} 22 \\ 58 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 7+18 \\ 21+45 \end{bmatrix} = \begin{bmatrix} 25 \\ 66 \end{bmatrix}$

$$= \left[\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 7 \\ 9 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 22 & 25 \\ 58 & 66 \end{bmatrix}$$

in general, $AB \neq BA$ ... if $AB = BA$, say that A and B <u>commute</u>

special case:

given $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

find $AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \left[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right]$

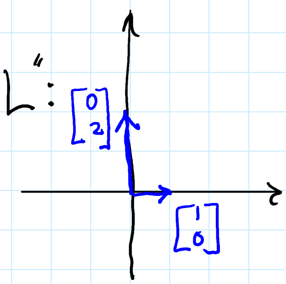
$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

find $BA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \left[\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right]$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -1 \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix}$$

$$BA \quad = \quad -AB$$

use "standard L":

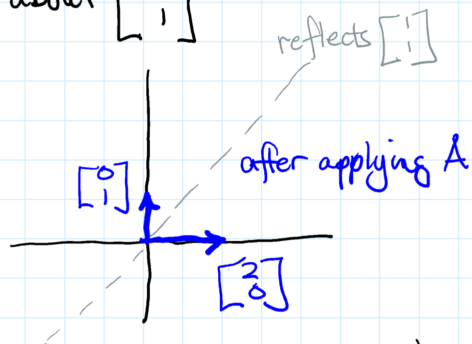


to interpret above multiplications geometrically

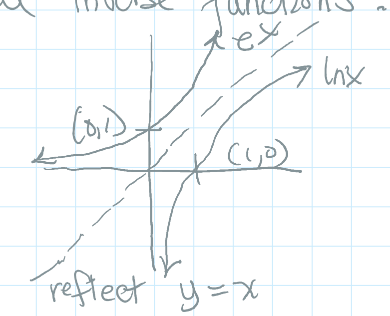
$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ reflects L about $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

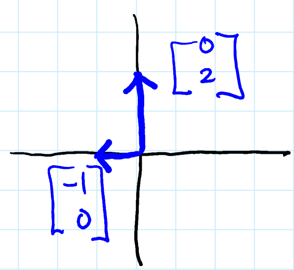
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$



recall inverse functions:

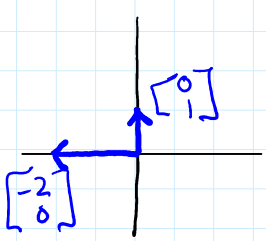


$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ negates the 1st vector



after applying (just) B

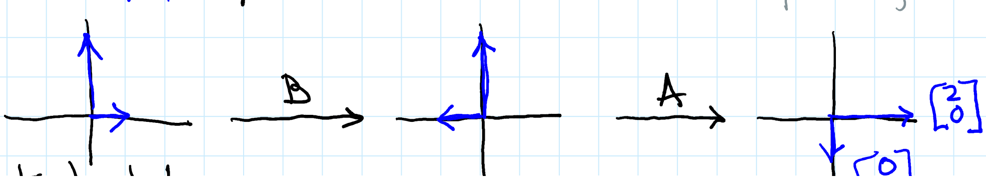
$A \longrightarrow B$



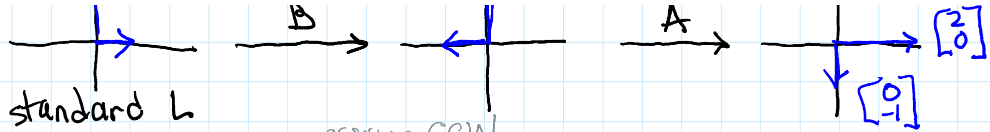
← this is BA

↑ like composition, work inside out

versus:



this is AB



this is AB

Conclusion: BA represents (assume CCW) rotation through $\frac{\pi}{2}$
AB " " " $-\frac{\pi}{2}$

More Matrix Algebra Rules

$x = 1x$

given $n \times m$ matrix A: $\quad AI_m = I_n A = A$

associativity: $(AB)C = A(BC) = ABC$

distributivity: given $\left.\begin{matrix} A, B & n \times p \\ C, D & p \times m \end{matrix}\right]$ matrices

then $A(C+D) = AC + AD$
$(A+B)C = AC + BC$

scalar: given $\left.\begin{matrix} A & n \times p \\ B & p \times m \end{matrix}\right]$ matrices then $(kA)B = A(kB) = k(AB)$
k is a scalar

Transition Matrix (revisited)

recall: square matrix is considered a transition matrix if all its columns are distribution vectors
↑ coeff. add to 1

recall: $\begin{matrix} 1 & \rightleftarrows & 2 \\ \downarrow & \swarrow & \uparrow \\ 3 & \rightarrow & 4 \end{matrix}$ ← system from L5/L6

found that the transition matrix [after 1 move] was

$$A = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

then $A(A\vec{x})$ would be the result after 2 moves

$= A^2 \vec{x}$

$\vdots$

then $A^m \vec{x}$ would be the result after m moves then as $m \to \infty$ the result would be 4 column vectors close to: $\begin{bmatrix} 1/6 \\ 1/3 \\ 1/4 \\ 1/4 \end{bmatrix}$ ← which was $\vec{x}_{equ}$ in original example

def'ns:

a transition matrix is <u>positive</u> if all entries are > 0

" is <u>regular</u> if all entries are <u>eventually</u> positive
i.e., A^m is positive for some $m \in \mathbb{Z}^+$

ex. $A = \begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix}$ <u>not</u> positive but it's regular:

$$A^2 = \begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix}$$

A^2 is positive

A is <u>regular</u> because A^2 is <u>positive</u>

Equilibria for Regular Transition Matrices

given A is a regular transition $n \times n$ matrix

a. then $\exists$ exactly one <u>distribution</u> vector $\vec{x}$ in $\mathbb{R}^n$ s.t. $A\vec{x} = \vec{x}$
i.e. $\vec{x}$ is $\vec{x}_{equ}$ and <u>all</u> its components are > 0

b. if $\vec{x}$ is any distribution vector in $\mathbb{R}^n$
then $\lim_{m \to \infty} (A^m \vec{x}) = \vec{x}_{equ}$.

c. $\lim_{m \to \infty} A^m = \begin{bmatrix} | & | & & | \\ \vec{x}_{equ} & \vec{x}_{equ} & \cdots & \vec{x}_{equ} \\ | & | & & | \end{bmatrix}$